การเพิ่มประสิทธิภาพของเสาสำหรับการรับแรงโก่งเดาะแบบอินอิลาสติกที่สูงที่สุด
Column Shape Optimization for Maximum Inelastic Buckling Capacity

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Abstract

This senior project demonstrates how inelastic behavior of a material affects the optimized-buckling-load shape of the column under volumetric constraint. Projected-gradient descent (PGD) and finite element method (FEM) is used to study the effect. The buckling load is calculated based on Euler-Bernoulli beam theory by finding the least eigenvalue from modified power method with Rayleigh quotient. For simplicity and due to the inelastic property of the material, bisection method is implemented for the iteration process to find the inelastic modulus corresponding to the critical buckling load of the column. The selected results show how inelastic modulus affects the shape and the optimized buckling load compare to linear elastic modulus case.

Keywords: projected gradient descent methods, finite element method, inelastic buckling, shape optimization.

1. Introduction

Optimization is becoming the essence of how engineers design their structures. Due to the limited resources and required conditions of the problems, engineers must come up with the efficient design that meet both the project performance and budget. Computer programming plays the important role on optimization method as the problem gets too complicated for human. Combining the sophisticated mind of the engineer and the computation power of the computer that keeps improving through time, we will be able to solve more daunting tasks that was thought to be unsolvable in the past.

Safety and cost have been the give and take the engineer must decide on the project. The safer of the structures, the higher of the price. There are many components and aspects in the building projects to be considered (e.g., capacity, serviceability etc.) as one wants the best cost-efficient out of it. The inelastic buckling of a column is chosen to investigate on this paper, because it contains the fundamental ideas in the structural engineering and takes a step further on the studying of the inelastic behavior of the material which tends to play an important role on real-world problems.

The problem of finding the strongest column was first solved mathematically by Keller [1] using the isoperimetric inequality. He found that the strongest column has an equilateral triangle as cross section, and it is tapered along its length, being thickest in the middle and thinnest at its ends with buckling load 61.2% larger than that of circular cylinder. For column with the same cross-section shape, the best tapering is found to increase the buckling load by one third over that of a uniform column. The work is further studied by Tadjbakhsh and Keller [2]. They used the isoperimetric inequality on the broader boundary conditions, i.e., clamped-clamped, clamped-pinned, pinned-pinned, and clamped-free boundary conditions. They found that the pinned-pinned column is the strongest column among the other cases. Then Olhoff and Rasmussen [3] reconsider the problem in the clamped-clamped case. They pointed out the necessary to consider the possibility that the optimum fundamental buckling load is a double eigenvalue, therefore, the governing equation is bimodal which indicates that the previous work [2] is not optimum. They also found that the work did not include the cross-section area constrains which allows internal hinges inside the column.

To resolve this issue, Wang, and Tada [4] made a reinvestigation on the prior research done by Tadjbakhsh and

Keller [2] and Olhoff and Rasmussen [3]. They concluded that problem of clamped-clamped column is indeed a double eigenvalue problem. Therefore, it is impossible to obtain the true optimum solution from single modal formulation that was previously assumed and done. They obtained two mutually symmetrical optimum eigenfunctions under numerical method which was incorrectly found to be asymmetry by Olhoff and Rasmussen [3]. Later, Hideyuki et al. [5] applied the Lagrange multiplier along with traction method, normally used in external work minimization problems, in dealing with the multimodal condition to find the optimum solution for critical buckling of a column-like and arch-like two-dimensional problems in a plane strain condition. Their result agrees with the work done by Tada and Wang [4]. Maalawi [6] focused on the practical fabrication aspect of the column due to the highly non-linear shape of the optimized column. The optimized shape also proven to be suboptimal due to the specific stiffness and mass distribution relation given into the column. To deal with the issue, Maalawi presented a practical discrete model, by considering a multi-segment column along with cross-sectional area, radius of gyration and length of each segment as designing variables. The model yields exact global optimal solution of clamped-clamped column.

Krishna, and Ram [7] studied on the discrete link-spring model of the column. They studied on regular cases where there are no internal vanishing cross-sectional areas in the column. Their work evaluates the strongest column for pinned-pinned boundary conditions from considering clamped-free or pinned-free conditions. Then their optimal system is determined recursively by using a one parameter iterative loop. Their model can also verify the optimal buckling load done by Tadjbakhsh and Keller [2]. Wang et al [8], developed Hencky bar-chain model (HBM) for the simple way to determine the buckling and vibration solution of non-uniform beams resting on partial variable elastic foundation. The HBM allows the analyst to replace complicated differential equation with set of linear algebraic equations which can be readily solved. HBM also has advantage on having clear physical meaning at each joint which is not always true for finite difference method (FDM) that requires introducing fictitious nodes.

Wang et al [9], detected a small error done by Krishna and Ram [7] on the assumption of using a small number of rotational springs to interpret the carrying column shape, which may produce an inaccurate higher buckling load of HBM than that of the continuum column because of the presence of rigid segments. Wang et al, also noted that the segmental number of HBM should be sufficiently large to capture the carrying shape of the continuum column. More importantly, the exact buckling solutions for the uniform HBM under axial load and self-weight as well as non-uniform HBM under axial load with specific class of spring stiffnesses are derived for the first time. Continuing their work, Wang et al. [10] proposed a simple but powerful method for optimization of inhomogeneous, elastically restrained columns against buckling subjected to both compressive concentrated and distributed axial loads that include self-weight. The proposed method adopted the Hencky bar-chain model for discretization of Euler columns. The critical buckling loads are found by finding the lowest eigenvalue of the system algebraic equation. The proposed optimization scheme is based on a parallel genetic algorithm. Comparing their work to the previously done solution shows the fast convergence, high accuracy, and flexibility of the proposed method.

The problem of finding the column shape for maximum elastic buckling capacity has been done by senior civil engineering student class of 2020. The use of gradient descent technique along with finite element method shows a very satisfying result due to its simplicity yet a powerful procedure for solving optimization problem dealing with critical buckling load of a column. However, in the real-world problems, materials often behave inelastically, thus using linear elastic material results in impractical solution. Introducing inelastic property of the material to the problem is expected to give the more realistic aspect to the problem and can be further adapted into more complex problems accounting inelastic behavior of the material.

The objective of this research is to study the effect of inelastic behavior of material on the shape and critical buckling load of an optimized clamped-free column subjected to an axially concentrated force at its free end under volumetric constrain.

2. Problem formulation

2.1 Formulation of the weak form

Consider a straight clamped-free column model under Euler beam theory, made of inelastic material subjected to axial force *P* at its free end. The cross-section of the column is circular throughout with area $A = A(x)$ where x is a global coordinate starts from the base of the column as $x = 0$ and $x = L$ at the column's free end. The column is subjected to the volumetric constraint

$$\int_0^L A(x)dx = V_0 \tag{1}$$

Where V_0 is a specified volume of material. The relationship between the bending moment *M*, the shear force *V*, the curvature *k*, the rotation θ, and the deflection *v* at any point of the column are given by

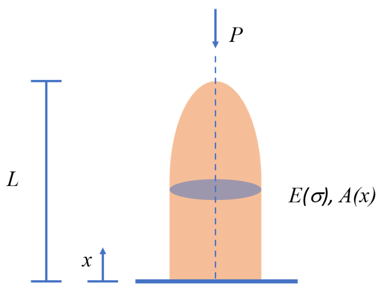


Figure 1 Model of cantilever column length L .

$$\frac{dM}{dx} + P_{cr} + \frac{dv}{dx} = V \tag{2}$$

$$M = E_t I \kappa \tag{3}$$

$$\frac{dV}{dx} = 0 \tag{4}$$

$$\kappa = \frac{d\theta}{dx}, \theta = \frac{dv}{dx} \tag{5}$$

where P_{cr} denotes the critical flexural buckling load of the column and E_t is the tangent modulus. The relationship between the moment of inertia I and cross-sectional area A , for the circular cross section, is defined as

$$I = \frac{A^2}{4\pi} \tag{6}$$

Substituting (3)-(6) into (2) leads to the equilibrium equation in terms of the deflection:

$$\frac{d^2}{dx^2}\left(\frac{E_t A^2}{4\pi}\frac{d^2 v}{dx^2}\right) + P_{cr}\frac{d^2 v}{dx^2} = 0 \tag{7}$$

In addition, the following boundary conditions at $x = 0$ and $x = L$ must be satisfied:

$$v(0) = 0, \theta(0) = 0, M(L) = 0, V(L) = 0 \tag{8}$$

Upon the normalizations, $\bar{x} = x / L$, $\bar{v} = v / L$, $\bar{A} = AL / V_0$, $\bar{E}_t = E_t / E_0$ and $\bar{P} = 4\pi P_{cr} L^4 / E_0 V_0^2$ the governing equation (7) and the volumetric constraint (1) become

$$\frac{d^2}{d\bar{x}^2}\left(\bar{E}_t \bar{A}^2 \frac{d^2\bar{v}}{d\bar{x}^2}\right) + \bar{P}\frac{d^2\bar{v}}{d\bar{x}^2} = 0 \tag{9}$$

$$\int_0^1 \bar{A}(x)d\bar{x} = 1 \tag{10}$$

For any given profile of the normalized cross-section area $\bar{A}$, the weak-form statement of the boundary value problem is established via a standard weighted residual technique together with the integration by parts and the enforcement of the boundary conditions (8). The final formulation is to find the normalized buckling load $\bar{P}$ such that there exists a nontrivial function $\bar{v} \in H_0^2$ such that

$$\int_0^1 \left[\bar{A}^2 \bar{E}_t \frac{d^2\bar{v}}{d\bar{x}^2}\frac{d^2\bar{w}}{d\bar{x}^2}\right] d\bar{x} - \bar{P}\int_0^1 \left[\frac{d\bar{v}}{d\bar{x}}\frac{d\bar{w}}{d\bar{x}}\right] d\bar{x} = 0 \quad \bar{w} \in H_0^2 \tag{11}$$

where

$$H_0^2 = \{f : f(0) = 0 \wedge f'(0) = 0 \wedge f'' \text{ is square integrable}\}$$

Finally, the statement of the corresponding optimization problem is to find the maximum buckling load of the column subjected to the volumetric constraint (10).

2.2 Material model

To take the material inelasticity into account, Ramberg-Osgood model is adopted in the present study. In particular, the stress-strain relationship is given by

$$\varepsilon = \frac{\sigma}{E_0} + K\left(\frac{\sigma}{E_0}\right)^{1/n_0} \tag{12}$$

where ε and σ denote the strain and stress, respectively; E_0 is the initial modulus; and K and $n_0 \le 1$ are model parameters. Note that the material parameters can be obtained via the calibration with experimental data. The normalized tangent modulus $\bar{E}_t$ for this particular model is given by

$$\bar{E}_t = \frac{E_t}{E_0} = \frac{1}{1 + \frac{K}{n_0}\left(\frac{\sigma}{E_0}\right)^{\frac{1}{n_0}-1}} \tag{13}$$

2.3 Discretization

Standard hermite shape, N^e ,is used to approximate the trial and test function.

$$N^e = \begin{Bmatrix} 1-3(\bar{x}^e/\bar{h}^e)^2+2(\bar{x}^e/\bar{h}^e)^3 \\ \bar{h}^e\{\bar{x}^e/\bar{h}^e - 2(\bar{x}^e/\bar{h}^e)^2+(\bar{x}^e/\bar{h}^e)^3\} \\ 3(\bar{x}^e/\bar{h}^e)^2-2(\bar{x}^e/\bar{h}^e)^3 \\ \bar{h}^e\{-(\bar{x}^e/\bar{h}^e)^2+(\bar{x}^e/\bar{h}^e)^3\} \end{Bmatrix} \tag{14}$$

Where $\bar{h}^e$ is length of the generic element Ω_e .Now, the trial and test functions are approximated by

$$\bar{v}^e(\bar{x}^e) = N^e u^e \tag{15}$$

$$\bar{w}^e(\bar{x}^e) = N^e w^e \tag{16}$$

where $\boldsymbol{u}^e = \{\bar{v}_1^e \quad \theta_1^e \quad \bar{v}_2^e \quad \theta_2^e\}^T$ is a vector containing the normalized displacement ($\bar{v}_1^e, \bar{v}_2^e$) and rotation (θ_1^e, θ_2^e) at both ends of the element and $\boldsymbol{w}^e$ is an arbitrary vector. The normalized area profile $\bar{A}$ over the generic element Ω_e is approximated by a constant function and denoted by $\bar{A}^e$. A collection of normalized areas for all elements is denoted by a vector $\bar{\boldsymbol{A}}$. Upon the approximations (15)-(16) and discretization of the normalized area profile together with the standard assembly procedure, the discretized form of (11) is given by

$$(K - PM)U = 0 \tag{17}$$

Where

$$K = \sum_{e=1}^{n} k^e, \quad k_{ij}^e = \bar{E}_t \bar{A}^2 \int_{\Omega_e} C^e (C^e)^T d\bar{x}^e \tag{18}$$

$$M = \sum_{e=1}^{n} m^e, \quad m^e = \int_{\Omega_e} B^e (B^e)^T d\bar{x}^e \tag{19}$$

With

$$B^e = dN^e / d\bar{x}^e, C^e = dB^e / d\bar{x}^e \tag{19}$$

$$\boldsymbol{k}^e = \frac{1}{(\bar{h}^e)^3}\begin{bmatrix} 12 & 6\bar{h}^e & -12 & 6\bar{h}^e \\ 6\bar{h}^e & 4(\bar{h}^e)^2 & -6\bar{h}^e & 2(\bar{h}^e)^2 \\ -12 & -6\bar{h}^e & 12 & -6\bar{h}^e \\ 6\bar{h}^e & 2(\bar{h}^e)^2 & -6\bar{h}^e & 4(\bar{h}^e)^2 \end{bmatrix} \tag{20}$$

$$\boldsymbol{m}^e = \frac{1}{30}\begin{bmatrix} -36/\bar{h}^e & -3 & 36/\bar{h}^e & -3 \\ -3 & -4\bar{h}^e & 3 & \bar{h}^e \\ 36/\bar{h}^e & 3 & -36/\bar{h}^e & 3 \\ -3 & \bar{h}^e & 3 & -4\bar{h}^e \end{bmatrix} \tag{21}$$

Upon the discretization of the normalized area profile via the piecewise constant function, the volumetric constraint (10) now becomes

$$\boldsymbol{F}\bar{\boldsymbol{A}} = 1 \tag{22}$$

where $\boldsymbol{F} = \bar{h}^e[1 \quad 1 \quad \cdots \quad 1]$

3. Solution scheme

After obtaining the eigenvalue system (17), the normalized critical buckling load $\bar{P}$ can be found by modified power method. The corresponding normalized tangent modulus, $\bar{E}_t$, can be found by bisection method with bracket answer.

After the normalized critical buckling load of the column for the given set of normalized cross-section area, $\bar{A}$, is found, the maximized critical buckling load is determined through Projected-Gradient-descent procedure. The process takes the initial guess of set of normalized area, $\bar{A}_0$ and iterates until the convergence value is lower than the specified tolerance.

4. Numerical result

4.1 Verification process

The normalized buckling load, $\bar{P}$, with increasing number of elements from n = 2 to 128 for the selected value, $n_0 = 0.8$ and $L^3 / V_0 = 0.001$, is reported in Table 1 and Figure 2. The value of $\bar{P}$ starts to converge when number of elements go beyond 64 elements. Figure 3 shows the cross-section area of the optimized column for the selected value. The graph also shows convergence of the area as the number of elements increase which agree with the convergence of normalized buckling load in Figure 2, therefore, the convergence of the numerical solution is confirmed.

The optimization procedure, projected gradient descent method, is verified using MATLABs built-in function, “fmincon”. The verifying result is reported in Table 2 for $n_0 = 0.8$ and $L^3 / V_0 = 0.001$. Comparing the critical buckling load, $\bar{P}$, obtain from each process shows smaller error between PDG and fmincon as the number of elements increase from 2 to 128 element. which confirmed the validity of the PDG method implemented into MATLABs.

Table 1 Comparing normalized Buckling ($\bar{\mathrm{P}}$) with varying number of elements (n) between PGD method and fmincon.

n	PGD Method	fmincon	Error(%) Compare with fmincon
2	0.657734	0.657734	11.69465926
4	0.705996	0.705996	5.215105474
8	0.729378	0.729377	2.075920698
16	0.739197	0.739189	0.75765376
32	0.742995	0.742971	0.247716557
64	0.744392	0.744391	0.060174453
128	0.744889	0.74484	0.006556235

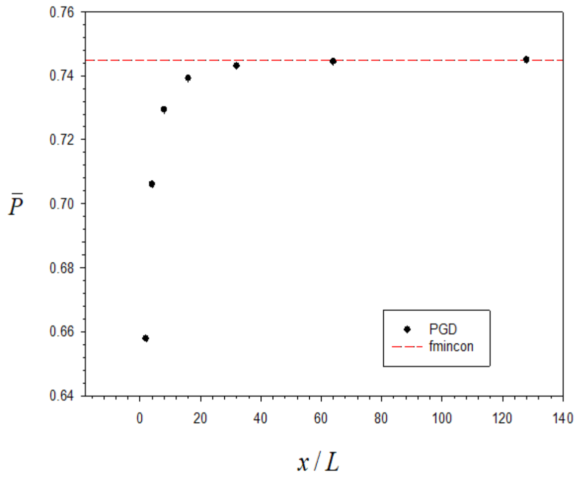


Figure 2 Normalized Buckling $\overline{\mathrm{P}}$ versus the number of elements

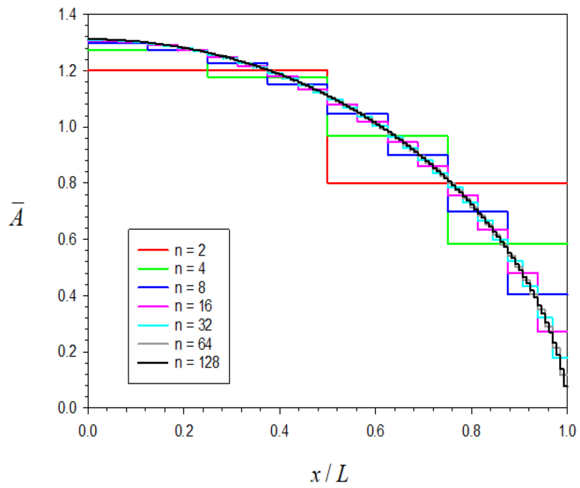


Figure 3: Normalized cross-section area $\overline{A}$ along the column as number of elements increase from 2 to 128.

4.2 Effect of inelasticity on normalized buckling load

Figure 4, in case of short column, the normalized buckling load increases as n_0 increases. The behavior starts to change as the column gets longer. For intermediate column, the relationship between normalized buckling load and n_0 is inverted such that the increasing n_0 no longer results in increasing buckling load but instead decreasing the nomalized buckling load. For very long column, n_0 will have very little effect on the normalized buckling load and the normalized buckling load tends to approache the same value as n_0 and L increase. This phenominon is directly related to the inelastic modulus model, Ramberg-Osgood's model, equation (13), where tangent modulus, E_t increases as n_0 increase when stress is high, as stress in short column, and decreases as n_0 increases when stress is low, as in long column. The value of tangent modulus also approach a single value as stress increases with variation of n_0 which gives the converging trend in figure 4 and figure 5.

4.3 Effect of inelasticity property on normalized cross-section

From figure 6 (a), short column, inelastic property of the material causes the area toward the base of the column to be smaller. The lower value of n_0, the smaller the areas. Howerver, for area toward the free end, the lower value of n_0 results in the bigger areas of the column. This gives the overall optimized column shape, for low value of n_0 ,to approach uniform shape. The main reason is directly related to Ramberg-Osgood model. Under the model, in low stress region or region toward column's base, lower value of n_0 gives higher tangent modulus. But in high stress region, region toward the column's free end, lower value of n_0 gives lower tangent modulus. With higher tangent modulus, the cross-section area can be smaller to resist the applied load while lower tangent modulus requires bigger area which cause the overall more uniform column shape as seen in figure 6 (a).

For longer length of column, the effect of n_0 on the cross-section area starts to diminish. Figure 6 (b-c) shows cross-section area for intermediate and long column. It can be clearly seen that the cross-section area of the column for converge to linear elastic case as column gets longer. Therefore, it can be concluded that as column gets longer, the inelastic property of the material has a little to none effect to both normalized critical buckling load and normalized cross-section area of the column.

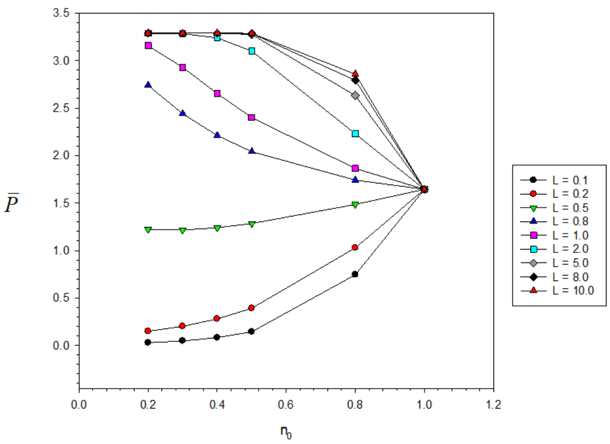


Figure 4 Normalized buckling load, $\overline{P}$ vs n_0 for different value of L

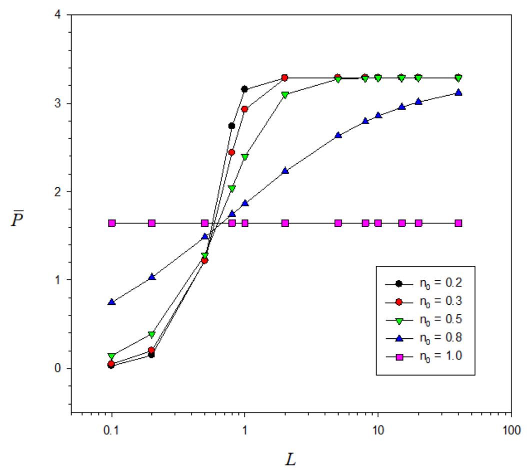
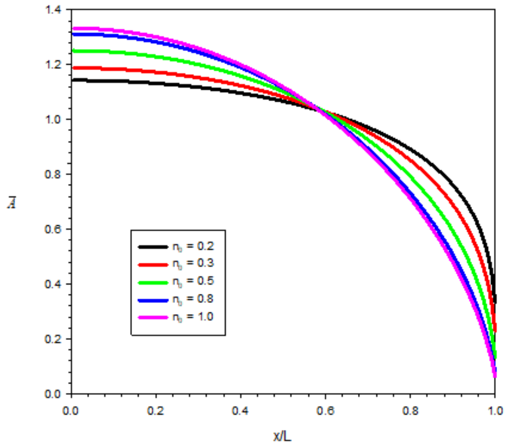


Figure 5 Normalized buckling load, $\overline{P}$ vs L for different value of n_0

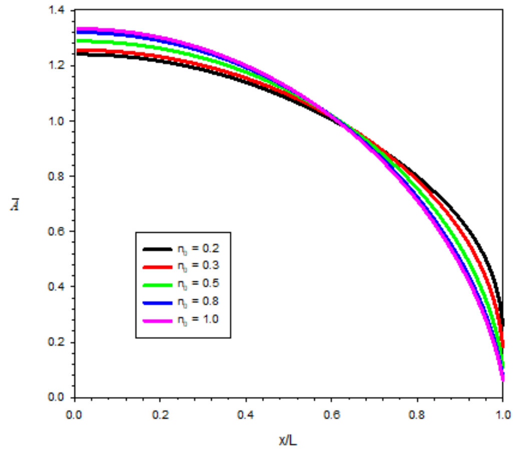
5. Conclusion

This paper studies on how inelastic property of material affects the optimized elastic buckling of column. Ramberg-Osgood model is used as inelastic model for the column. Projected gradient descent method is used for optimization process along with power method and bisection method.

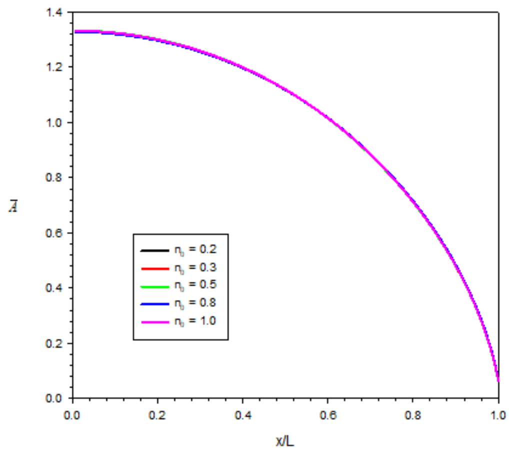
The result indicates that inelastic property mainly affects short column for both shape and the critical buckling load. With more inelastically material, short column tends to decrease its base area and increase its free end area, this result in more uniform optimized column compared to optimized elastic column. The effect of inelastic property is lower as the length of the column is longer and column shape converge to linear elastic optimized shape regardless of inelastic property as the length increases. Additionally, the normalized buckling load of short column increases as the material behaves more inelastically. The trend is inversed as the column transfer from short to long column, results in the decreasing normalized buckling load as material behaves more inelastically. Finally, for n_0 approaching 1.0 and longer column, the shape and normalized buckling load of the column approach a single value which is the linear elastic case.



(a)



(b)



(c)

Figure 6 Normalized cross-section area, $\overline{A}$ along the column for (a) short column, (b) intermediate column, and (c) long column

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